

The unimodality of the Ehrhart δ -polynomial of the chain polytope of the zig-zag poset

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Abstract. We prove the unimodality of the Ehrhart δ -polynomial of the chain polytope of the zig-zag poset, which was conjectured by Kirillov. First, based on a result due to Stanley, we show that this polynomial coincides with the W -polynomial for the zig-zag poset with some natural labeling. Then, its unimodality immediately follows from a result of Gasharov, which states that the W -polynomials of naturally labeled graded posets of rank 1 or 2 are unimodal.

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1 Introduction

The main objective of this paper is to prove a unimodality conjecture on δ -polynomials of the chain polytope of the zig-zag poset, which was proposed by Kirillov [5] in the study of Kostka numbers and Catalan numbers. Let us first give an overview of Kirillov's conjecture.

Let \mathbb{Z}^m denote the m -dimensional integer lattice in \mathbb{R}^m , and let \mathcal{P} be an m -dimensional lattice polytope in \mathbb{R}^m . A remarkable theorem due to Ehrhart [3] states that the number of lattice points that lie inside the dilated polytope $n\mathcal{P}$:

$$i(\mathcal{P}; n) = |n\mathcal{P} \cap \mathbb{Z}^m|. \quad (1.1)$$

is given by a polynomial in n of degree m , called the Ehrhart polynomial of the lattice polytope \mathcal{P} . By a well known result about rational generating functions, see [9, Corollary 4.3.1], the generating function (called the Ehrhart series of \mathcal{P})

$$J(\mathcal{P}; t) = \sum_{n \geq 0} i(\mathcal{P}; n) t^n \quad (1.2)$$

evaluates to a rational function:

$$J(\mathcal{P}; t) = \frac{\delta(\mathcal{P}; t)}{(1 - t)^{\dim \mathcal{P} + 1}} \quad (1.3)$$

for some polynomial $\delta(\mathcal{P}; t)$ of degree at most $\dim(\mathcal{P})$, which is called the Ehrhart δ -polynomial of \mathcal{P} . If the polynomial $\delta(\mathcal{P}; t)$ is of the following form

$$\delta(\mathcal{P}; t) = \delta_0 + \delta_1 x + \cdots + \delta_m x^m,$$

then we call $(\delta_0, \delta_1, \dots, \delta_m)$ the (Ehrhart) δ -vector of \mathcal{P} . Stanley [7] also proved that $\delta(\mathcal{P}; t)$ must be a polynomial in nonnegative coefficients. For more information on the Ehrhart theory of rational polytopes, see [1].

Let \mathcal{P}_n be a convex integral polytope in \mathbb{R}^n determined by the following inequalities

$$\begin{aligned} x_i &\geq 0, & \text{for } 1 \leq i \leq n, \\ x_i + x_{i+1} &\leq 1, & \text{for } 1 \leq i \leq n-1. \end{aligned}$$

Kirillov conjectured that the δ -polynomial of \mathcal{P}_n is unimodal [5]. Recall that a polynomial $f(x) = \sum_{i=0}^n a_i x^i$ with real coefficients is said to be unimodal if there exists an integer $i \geq 0$ such that

$$a_0 \leq \cdots \leq a_{i-1} \leq a_i \geq a_{i+1} \geq \cdots \geq a_n,$$

and symmetric if for all $0 \leq i \leq n$

$$a_i = a_{n-i}.$$

Kirillov's conjecture is stated as follows.

Conjecture 1.1 ([5, p.119, Conjecture 3.11]) *For any $n \geq 1$, the δ -polynomial $\delta(\mathcal{P}_n; t)$ is unimodal.*

In this paper, we give a proof of Kirillov's conjecture. Our proof is based on the theory of chain polytopes of posets, as well as the theory of W -polynomials of posets.

2 Preliminaries

In this section, we shall review some definitions and results on chain polytopes and W -polynomials of posets.

We begin with some definitions concerning posets. Let (P, \preceq) be a poset with d elements. Recall that a chain of length ℓ in P is a sequence $a_0 \prec a_1 \prec$

$\cdots \prec a_\ell$, and it is called maximal in P if we cannot add elements to this chain. If every maximal chain of P has the same length r , then we say that P is graded of rank r and denote the rank of P by $\text{rank}(P)$. In this case, there is a unique rank function $\rho : P \rightarrow \{0, 1, \dots, d\}$ such that $\rho(x) = 0$ if x is a minimal element of P , and $\rho(y) = \rho(x) + 1$ if y covers x in P . If $\rho(x) = i$, then we say that x is of rank i .

The notion of chain polytopes was introduced by Stanley [8]. Given a poset P with elements $\{a_1, \dots, a_d\}$, Stanley associated it with a polytope $\mathcal{C}(P)$ defined by the chains in P , called the chain polytope of P . Precisely, the chain polytope $\mathcal{C}(P)$ is the convex polytope consisting of those $(x_1, \dots, x_d) \in \mathbb{R}^d$ such that

- $x_i \geq 0$, for every $a_i \in P$,
- $x_{p_1} + x_{p_2} + \cdots + x_{p_k} \leq 1$, for every chain $a_{p_1} \prec \cdots \prec a_{p_k}$ of P .

Since $\mathcal{C}(P)$ contains the d -dimensional simplex

$$\{(x_1, \dots, x_d) \in \mathbb{R}^d : x_i \geq 0 \text{ for all } 1 \leq i \leq d \text{ and } x_1 + x_2 + \cdots + x_d \leq 1\},$$

we know that

$$\dim(\mathcal{C}(P)) = d = |P|.$$

We would like to point out that the polytope \mathcal{P}_n is just the chain polytope of the zig-zag poset of order n . Recall that the zig-zag poset of order n is the poset $\mathbf{Z}_n = \{a_1, \dots, a_n\}$ in which

$$a_1 \prec a_2 \succ a_3 \prec \cdots,$$

see [2, 9]. Note that, if $n = 2k + 1$, the maximal chains of \mathbf{Z}_n are $a_{2i-1} \prec a_{2i}$ and $a_{2i+1} \prec a_{2i}$ for $1 \leq i \leq k$. While, if $n = 2k + 2$, the maximal chains of \mathbf{Z}_n are $a_{2i-1} \prec a_{2i}$ and $a_{2i+1} \prec a_{2i}$ for $1 \leq i \leq k$, together with $a_{2k+1} \prec a_{2k+2}$. By definition, it is clear that

$$\mathcal{P}_n = \mathcal{C}(\mathbf{Z}_n). \tag{2.1}$$

Based on the above viewpoint, Conjecture 1.1 is equivalent to the statement that the δ -polynomial of $\mathcal{C}(\mathbf{Z}_n)$ is unimodal. While for the chain polytope of poset P , Stanley [8] has already established a connection between the δ -polynomial of the chain polytope and the number of order-preserving maps of P . Let m be a positive integer and let $\tilde{\Omega}(P; m)$ denote the number of order-preserving maps $\eta : P \rightarrow \{1, 2, \dots, m\}$, i.e., if $x \preceq y$ in P then

$\eta(x) \leq \eta(y)$. It is known that $\tilde{\Omega}(P; m)$ is a polynomial of degree $|P|$ in m . Equivalently, there exists a polynomial $\tilde{W}(P; t)$ of degree $\leq |P|$ such that

$$\sum_{m \geq 0} \tilde{\Omega}(P; m+1)t^m = \frac{\tilde{W}(P; t)}{(1-t)^{|P|+1}}. \quad (2.2)$$

Stanley obtained the following theorem.

Theorem 2.1 ([8, Theorem 4.1]) *For any positive integer m and any poset P , we have*

$$i(\mathcal{C}(P); m) = \tilde{\Omega}(P; m+1),$$

or equivalently,

$$\delta(\mathcal{C}(P); t) = \tilde{W}(P; t). \quad (2.3)$$

Instead of considering the number of order-preserving maps of P , we may also study the number of order-reversing maps of P . In fact, there is a more general theory on order-reversing maps, developed by Stanley [6] and called the theory of P -partitions. Suppose that P is a finite poset with d elements as before. A labeling ω of P is a bijection from P to $\{1, 2, \dots, d\}$. The labeling ω is called natural if $x \preceq y$ implies $\omega(x) \leq \omega(y)$ for any $x, y \in P$, namely, it is an order-preserving map. A (P, ω) -partition is a map σ which satisfies the following conditions:

- σ is order reversing, namely, $\sigma(x) \geq \sigma(y)$ if $x \preceq y$ in P ; and moreover
- if $\omega(x) > \omega(y)$, then $\sigma(x) > \sigma(y)$.

The order polynomial $\Omega(P, \omega; n)$ is defined as the number of (P, ω) -partitions σ with $\sigma(x) \leq n$ for any $x \in P$. It is also known that $\Omega(P, \omega; n)$ is a polynomial of degree $|P|$ in n , or equivalently, there exists a polynomial $W(P, \omega; t)$, called the W -polynomial of (P, ω) , of degree $\leq |P|$ such that

$$\sum_{n \geq 0} \Omega(P, \omega; n+1)t^n = \frac{W(P, \omega; t)}{(1-t)^{|P|+1}}. \quad (2.4)$$

Note that, for a natural labeling ω , we must have

$$\Omega(P, \omega; n) = \tilde{\Omega}(P; n), \quad (2.5)$$

since $\Omega(P, \omega; n)$ is just the number of order-reversing maps in this case, and $\tilde{\Omega}(P; n)$ is the number of order-preserving maps. In fact, if $\eta : P \rightarrow$

$\{1, 2, \dots, m\}$ is order reversing, then the map $\tilde{\eta} : P \rightarrow \{1, 2, \dots, m\}$ defined by

$$\tilde{\eta}(x) = m + 1 - \eta(x)$$

is order-preserving, and vice versa. By (2.3) and (2.5), we have

$$\delta(\mathcal{C}(P); t) = \widetilde{W}(P; t) = W(P, \omega; t). \quad (2.6)$$

3 Proof

In this section, we shall give a proof of Conjecture 1.1. Our proof is based on the following result due to Gasharov [4].

Theorem 3.1 ([4, Theorem 1.2]) *If P is a graded poset with $1 \leq \text{rank}(P) \leq 2$ and ω is a natural labeling of P , then $W(P, \omega; t)$ is unimodal.*

We proceed to prove Conjecture 1.1.

Proof of Conjecture 1.1. By (2.6), we have

$$\delta(\mathcal{C}(\mathbf{Z}_n); t) = \widetilde{W}(\mathbf{Z}_n; t) = W(\mathbf{Z}_n, \omega; t)$$

for some natural labeling ω of the zig-zag poset \mathbf{Z}_n . It is clear that \mathbf{Z}_n a graded poset with $\text{rank}(P) = 1$. From Theorem 3.1 it follows the unimodality of $\delta(\mathcal{C}(\mathbf{Z}_n); t)$, and hence that of $\delta(\mathcal{P}_n; t)$. This completes the proof. ■

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References

- [1] M. Beck and S. Robins, *Computing the Continuous Discretely*, Undergraduate Texts in Mathematics, Springer, New York, 2007.
- [2] T. S. Blyth and J. C. Varlet, *Ockham Algebras*, Oxford Science Publications, Oxford Univ. Press, New York, 1994.
- [3] E. Ehrhart, Sur les polyèdres rationnels homothétiques à n dimensions, C. R. Acad. Sci. Paris **254** (1962), 616–618.
- [4] V. Gasharov, On the Neggers-Stanley conjecture and the Eulerian polynomials, J. Combin. Theory Ser. A **82** (1998), no. 2, 134–146.

- [5] A. N. Kirillov, Ubiquity of Kostka polynomials, in *Physics and combinatorics 1999 (Nagoya)*, 85–200, World Sci. Publ., River Edge, NJ.
- [6] R. P. Stanley, *Ordered Structures and Partitions*, Amer. Math. Soc., Providence, RI, 1972.
- [7] R. P. Stanley, Decompositions of rational convex polytopes, *Ann. Discrete Math.* **6** (1980), 333–342.
- [8] R. P. Stanley, Two poset polytopes, *Discrete Comput. Geom.* **1** (1986), no. 1, 9–23.
- [9] R. P. Stanley, *Enumerative Combinatorics. Vol. 1*, Cambridge Studies in Advanced Mathematics, 49, Cambridge Univ. Press, Cambridge, 1997.